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A note on the conservation of the axial momentum of a turbulent jet

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The conservation law for the flux of axial momentum in a turbulent jet is examined. The examination discloses that for a plane jet out of a wall the momentum flux is reduced appreciably because the induced flow towards the jet has a component in the direction opposite to the main jet flow and because of the pressure field generated in the ambient fluid. Existing experimental results confirm this conclusion.

1. Introduction

Previous investigators, including Townsend (1976), Albertson *et al.* (1950) and Schlichting (1960), apparently believed that the momentum flux in any jet is very nearly constant. In this investigation the conservation equation for the flux of momentum in a turbulent jet is reformulated and used to show that there are realistic circumstances (e.g. in a plane jet out of a wall) where the constancy of momentum flux is an inaccurate approximation.

2. Basic definitions and concepts

A plane jet is defined as a source of fluxes of mass $\rho\mu_0$ and momentum ρm_0 (per unit span) through a slot of thickness D into a finite space filled with fluid of density ρ. It is assumed that the jet is turbulent and that the ambient fluid is quiescent, except for flows induced by the presence of the jet. The flow regime in a large container of characteristic scale $L \gg D$ may be depicted as in figure 1 and clearly depends on the shape of the container. However, the flow in some region of characteristic length scale l, where $D \ll l \ll L$ (figure 1), may be assumed independent of the geometry at a distance L. It is the flow in this region (the jet and the neighbouring induced flow) which is investigated.

Instantaneously an irregular and sharp interface (called the turbulent interface) separates the fluid which is turbulent from the ambient fluid, which may be assumed irrotational (see Corrsin & Kistler 1954). The jump in the instantaneous velocity field across the turbulent interface is of order V, where V is some characteristic velocity of the turbulent jet. The thickness of the turbulent interface is of order ν/V (Saffman 1970), where ν is the kinematic viscosity. If the jet carries a tracer (dye, salt, etc.) a sharp concentration interface divides the jet fluid from the ambient fluid. By analogy, the thickness of this interface is of order K/V, where K is the molecular diffusivity of the tracer. The thicknesses of both the turbulent and the concentration interface are several orders of magnitude smaller than some characteristic length scale b of the jet

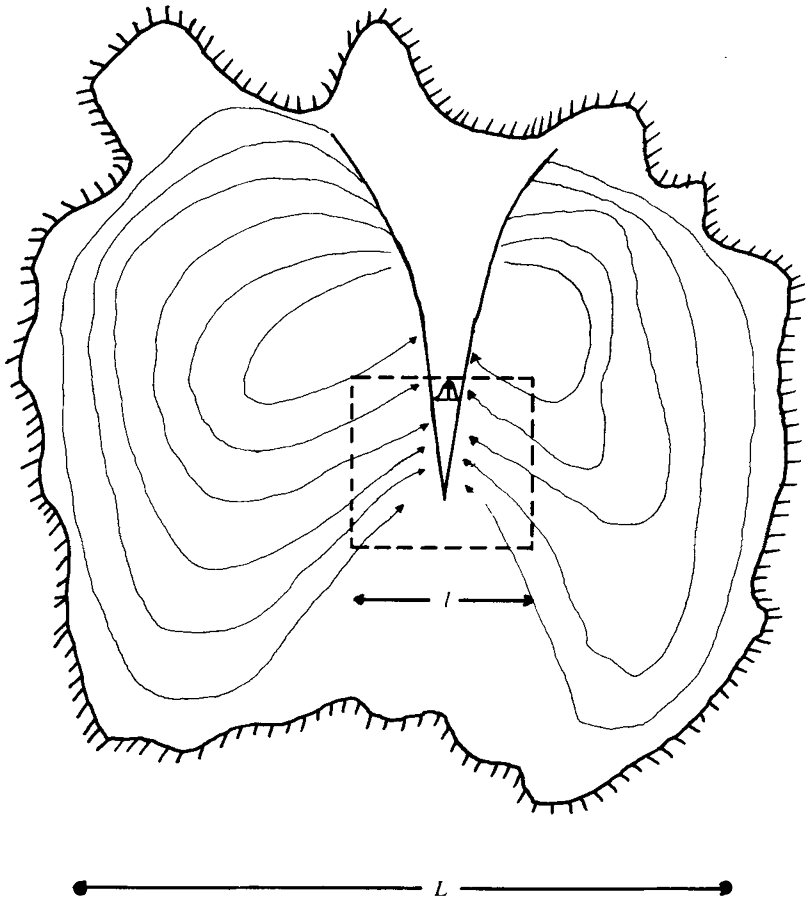

FIGURE 1. Hypothetical flow regime of a jet in a finite enclosure.

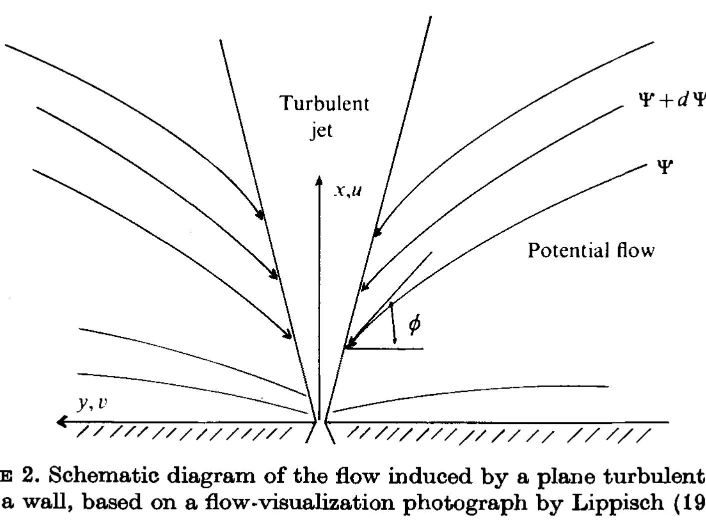

FIGURE 2. Schematic diagram of the flow induced by a plane turbulent jet out of a wall, based on a flow-visualization photograph by Lippisch (1958).

cross-section and therefore it can be assumed (even if $\nu \neq K$) that at any instant the two interfaces coincide. This implies that photographs of coloured jets can be used to distinguish regions of turbulent jet fluid from the ambient fluid.

In a time-exposure photograph of a jet, a fictitious but fairly well defined, time-independent interface appears which separates a region that is sometimes occupied by turbulent jet fluid from the ambient fluid. This fictitious interface will be called the 'jet boundary' in this paper because it bounds the region of turbulent mixing. The jet boundary can be viewed as the envelope of the instantaneous turbulent (or concentration) interfaces. A good photograph of a plane jet out of a wall was published by Lippisch (1958). On the basis of this figure, a schematic flow diagram such as figure 2 can be drawn which indicates the turbulent region, the jet boundaries and the streamlines $\Psi(x,y)$ of the outer flow (see also Kotsovinos 1975). The jet boundaries can be described by the equation $y = \pm B(x)$, where $B(x)$ can be estimated from time-exposure photographs. Next the conservation equation of momentum is examined, keeping in mind figure 2.

3. Conservation equation for the flux of axial momentum

The continuity and axial momentum equations can be integrated across the jet from $-B(x)$ to $B(x)$ to obtain the conservation equation for the volume flux (the symbols are defined in figure 2),

$$\frac{d}{dx}\int_{-B(x)}^{B(x)} \bar{u}(x,y)\,dy = -2\bar{v}(x,B(x)) + 2(dB(x)/dx)\,\bar{u}(x,B(x)) \tag{1}$$

(see Kotsovinos 1978), and the conservation equation for the axial momentum,

$$\frac{d}{dx}\int_{-B(x)}^{B(x)} (\bar{u}^2 + \overline{u'^2} + \bar{p}/\rho)\,dy = -\left[\bar{u}\bar{v} + \overline{u'v'}\right]_{-B(x)}^{B(x)} + 2(dB(x)/dx)\,\{\bar{u}^2(x,B(x)) + \overline{u'^2}(x,B(x)) + \bar{p}(x,B(x))/\rho\}. \tag{2}$$

The axial momentum equation (2) is integrated from $x = 0$ to x to obtain

$$m(x) = \int_{-B(x)}^{B(x)} (\bar{u}^2 + \overline{u'^2} + \bar{p}/\rho)\,dy = m_0 + C(x) + H(x),$$

where

$$C(x) = 2\int_0^x [-\bar{u}(x,B(x))\,\bar{v}(x,B(x)) - \overline{u'v'}(x,B(x)) + (dB(x)/dx)\,\{\bar{u}^2(x,B(x)) + \overline{u'^2}(x,B(x))\}]\,dx$$

and

$$\rho H(x) = \int_0^x 2(dB(x)/dx)\,\bar{p}(x,B(x))\,dx.$$

Previous investigators (including Schlichting 1960, p. 606; Townsend 1976, p. 193; Tennekes & Lumley 1972, p. 112) assumed, explicitly or implicitly, that a good approximation is to assume that $H(x) = C(x) = 0$ for any x so that

$$m(x) = \int_{\text{jet}} (\bar{u}^2 + \overline{u'^2} + \bar{p}/\rho)\,dy = m_0. \tag{3}$$

Equation (3) (which states the constancy of the total flow force $\rho m(x)$ across the jet) plays a major role in understanding and explaining the basic features of the jet mechanics. However, there is a tendency to take (3) as a universal law and use it for calibration (Flora & Goldschmidt 1969) or as a test for good experimental results. It is the objective of this paper to show that this tendency is in general unjustified and that there are realistic cases where the terms $H(x)$ and $C(x)$ cannot be neglected. Now the magnitude of these terms is estimated.

The analysis of Phillips (1955) and Stewart (1956) and the experimental results of Bradbury (1965) can be used to show that for a plane jet out of a wall the terms $\overline{u'^2}$ and $\overline{u'v'}(x, B(x))$ (representing fluctuations of the induced flow field) are negligible in comparison with the time-averaged terms $\overline{u}^2(x, B(x))$ and $\overline{u}(x, B(x))\,\overline{v}(x, B(x))$.

The time-averaged mean pressure along the (fictitious) jet boundaries $y = \pm B(x)$ can be calculated from the Bernoulli theorem, since the outer flow can be assumed irrotational (Corrsin & Kistler 1954; Stewart 1956):

$$\tfrac{1}{2}\{\overline{u}^2(x, B(x)) + \overline{v}^2(x, B(x))\} + \overline{p}(x, B(x))/\rho = 0$$

(where $\overline{p}$ is the pressure relative to the hydrostatic pressure). Assuming that

$$\overline{u}(x, B(x)) = \overline{v}(x, B(x)) \tan\phi(x)$$

(see figure 2) and using (1) gives

$$\overline{p}(x, B(x)) = \rho(1+\tan^2\phi)\,(d\mu/dx)^2/8(-1+k\tan\phi)^2,$$

where $k = dB(x)/dx$.

It is therefore apparent that

$$C(x) \approx \int_0^x \frac{(d\mu/dx)^2 \tan\phi(x)}{2(-1+k\tan\phi(x))}\,dx \tag{4}$$

and

$$H(x) \approx -\int_0^x \frac{k(1+\tan^2\phi)\,(d\mu/dx)^2}{4(-1+k\tan\phi)^2}\,dx. \tag{5}$$

It is clear that the term $C(x)$ (due to the induced velocity field) can be either positive or negative, depending on the angle ϕ (which in turn depends on the existence of solid boundaries close to the jet). However, the term $H(x)$ (due to the ambient pressure) is always negative. Clearly the parameters that should be specified in order to integrate numerically or analytically the functions $C(x)$ and $H(x)$ are the angle $\phi(x)$, the curvature k of the jet boundary and the volume flux $\mu(x)$. In the region of flow establishment ($x < 6D$) the experimental results of Liepmann & Laufer (1947) can be used to obtain

$$d\mu(x)/dx = 0{\cdot}064\,\overline{u}_0 = 0{\cdot}064(m_0/D)^{\frac{1}{2}},$$

where $\overline{u}_0$ is the velocity at the exit of the jet. For $x > 6D$ the jet is fully developed and the increase in volume flux $d\mu(x)/dx$ can be assumed to be of the form

$$d\mu(x)/dx = 0{\cdot}28 m_0^{\frac{1}{2}} x^{-\frac{1}{2}}. \tag{6}$$

The angle $\phi(x)$ depends on the solid boundaries close to the jet (see Taylor 1958). In this study the special case of a plane jet out of a wall is examined. The flow visualization in figure 2 is used to get rough estimates of the angle $\phi(x)$. It is recognized that more detailed visualization (or experimental studies in general) of the induced flow field is needed for better estimates. For simplicity it is assumed as a fair approximation that $\phi(x)$ is constant at a value $\frac{1}{4}\pi$ for values of x larger than $6D$ and that for $0 < x < 6D$ the angle $\phi(x)$ increases linearly from zero to $\frac{1}{4}\pi$. It is also estimated that $B(x) \approx \pm 0{\cdot}25x$, i.e. $k \approx \pm 0{\cdot}25$. It is then easy to integrate (4) and (5) to find that for $x > 6D$

$$m(x)/m_0 \approx 0{\cdot}983 - 0{\cdot}0693 \ln (x/6D). \tag{7}$$

The important conclusion from (7) is that the flow force $\rho m(x)$ is not constant but decreases as the distance x increases. For example, (7) predicts that at $x = 100D$, $m(x)/m_0 \approx 0{\cdot}8$. It is recognized that the coefficients which appear in (7) have been derived from estimates of the parameters $\phi(x)$, $B(x)$ and $d\mu(x)/dx$. However, these estimates are good enough to support the point of this study: that (3) is not a valid approximation for a plane jet out of a wall. A comparison with experimental results is presented in the next section.

Equation (6) [which has been used to derive (7)] has been derived from dimensional analysis under the assumption that the increase in volume flux (entrainment) is dependent on the initial momentum flux ρm_0 and the distance x but independent of the induced flow and the ambient pressure. Therefore (6) and (7) should be viewed as valid approximations for values of x of the order of a few thousand slot thicknesses.

4. Experimental results

In this section existing experimental results will be used to calculate the flow force $\rho m(x)$ at various distances x. Miller & Comings (1957) and Bradbury (1965) found experimentally that the distributions of $\overline{u}'$ and $\overline{p}/\rho$ across the jet are of the same order and of opposite sign, so that it seems a very good approximation to assume that

$$\int_{-B(x)}^{B(x)} (\overline{u'^2} + \overline{p}/\rho)\, dy/m(x) \approx 0,$$

which implies that

$$m(x) = \int_{-B(x)}^{B(x)} \overline{u}^2(x,y)\, dy, \tag{8}$$

i.e. that the jet flow force is equal to the momentum flux of the mean flow across the jet (usually called the 'momentum flux').

Other experimental studies (Miller 1957; Goldschmidt & Eskinazi 1966; Reichardt 1942; Van der Hegge Zijnen 1957) showed that the cross-sectional distribution of the time-averaged mean axial velocity $\overline{u}(x,y)$ can be very well described in the region $y < |0{\cdot}17x|$ by a Gaussian curve of the form

$$\overline{u}(x,y) = \overline{u}_m(x) \exp\{-\ln [2(y/b(x))^2]\},$$

where $\overline{u}_m(x) = \overline{u}(x,0)$ and where $b(x)$ is the velocity half-width defined by the relation $\overline{u}(x, \pm b(x)) = \frac{1}{2}\overline{u}_m(x)$. For $y \gtrsim |0{\cdot}17x|$ the experimental data deviate from the Gaussian curve. For the specific case of a plane jet out of a wall (figure 2) the axial velocity along the jet boundaries should be negative (if the velocity along the jet axis is defined as positive), so that the velocity profile should, ideally, be as in figure 3. It is apparent that the location where $\overline{u}(x,y) = 0$ is inside the turbulent region. The existing experimental results on the mean velocity profile are of limited value for (approximately) $y > |0{\cdot}17x|$ because the Pitot tube and the hot wire are inadequate instruments when flow reversal occurs.

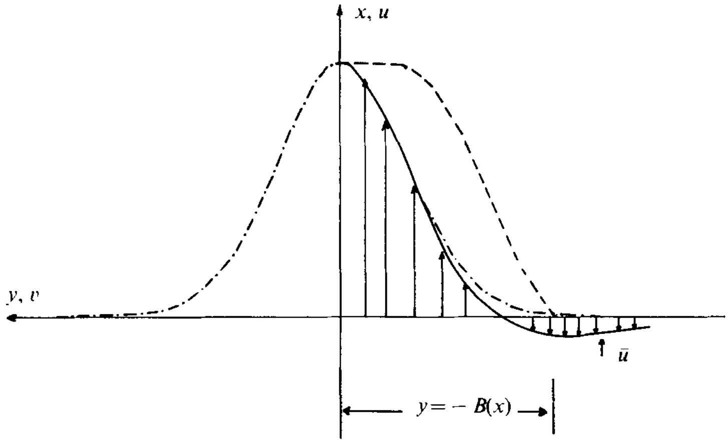

FIGURE 3. Sketch of the 'true' velocity and intermittency profiles in a plane turbulent jet out of a wall. ——, $\overline{u}(x,y)/\overline{u}_m$; —·—, Gaussian approximation; – – –, intermittency factor.

Fortunately, these experimental difficulties are not important for this investigation because the axial velocity profile in the region $y < |0{\cdot}17x|$ (where experimental data of high accuracy are obtained) contributes more than 99 % of the axial momentum flux $\rho m(x)$, as defined by (8). In order to demonstrate this, it will be assumed for the moment (and without loss of generality) that $b(x) = 0{\cdot}1x$. It is then easy to show that

$$\int_{-0\cdot17x}^{0\cdot17x} \overline{u}^2(x,y)\,dy = \int_{-1\cdot7b(x)}^{1\cdot7b(x)} \overline{u}_m^2(x)\exp\{-2\ln[2(y/b(x))^2]\}\,dy$$
$$= 0{\cdot}997(\pi/2\ln 2)^{\frac{1}{2}}\,b(x)\,\overline{u}_m^2(x), \quad (9)$$

i.e. the kinematic momentum flux $m(x)$ can be calculated fairly accurately from experimental data using the relation

$$m(x) = (\pi/2\ln 2)^{\frac{1}{2}}\,b(x)\,\overline{u}^2(x). \quad (10)$$

The initial momentum flux of a plane jet is

$$\rho m_0 = \rho A\,\overline{u}_0^2 D,$$

where $\overline{u}_0 = \overline{u}(0,0)$ and A is a constant which depends on the exact shape of the velocity profile at $x = 0$. For any particular group of measurements, A is a constant. For a uniform initial velocity profile $A = 1$. In general,

$$m(x)/m_0 = A^{-1}(\pi/2\ln 2)^{\frac{1}{2}}\,(\overline{u}_m^2(x)/\overline{u}_0^2)\,b(x)/D. \quad (11)$$

If the momentum flux $m(x)$ is a constant, then the ratio $Am(x)/m_0$ should be a constant for any x. This ratio, calculated from experimental results presented in the literature, is tabulated in table 1 and plotted in figure 4 *vs*. the distance x/D from the jet exit. The interesting conclusion is that the ratio $Am(x)/m_0$ for a particular group of measurements decreases with increasing x/D, which implies that the momentum flux $m(x)$ in a plane jet out of a wall is not constant. The experimental results of Heskestad (1965) and of Kotsovinos (1975) agree quite well with the prediction from (7) for $A \approx 0{\cdot}88$.

Since the constancy of momentum flux has been viewed as an exact and unquestion able statement by many investigators, it is possible that some of them used 'the constancy of momentum' to calibrate their Pitot tubes (see Flora & Goldschmidt

Investigator	x/D	b/D	$\bar{u}_m^2/\bar{u}_0^2$	$Am(x)/m_0$
Miller (1957)	10	1·11	0·540	0·903
	20	2·07	0·289	0·900
	30	3·03	0·184	0·84
	40	3·99	0·144	0·86
Goldschmidt & Eskinazi (1966)	25	2·40	0·252	0·91
	30	2·89	0·20	0·87
	35	3·38	0·165	0·84
	40	3·87	0·141	0·82
	45	4·37	0·123	0·81
	50	4·86	0·109	0·80
	60	5·84	0·089	0·78
Heskestad (1965)	47	4·71	0·105	0·744
	65	6·69	0·071	0·710
	85	8·89	0·052	0·692
	103	10·87	0·049	0·680
	125	13·29	0·034	0·674
	155	16·59	0·027	0·666
Kotsovinos (1975)	14	1·8	0·337	0·91
	20·8	2·14	0·275	0·89
	33.15	3·33	0·169	0·85
	37·60	3·76	0·145	0·82
	37·0	3·54	0·152	0·81
	54·17	5·42	0·092	0·75
	58·75	5·96	0·084	0·75
	74·16	7·08	0·066	0·70

TABLE 1. Experimental data. The last column is calculated from equation (11).

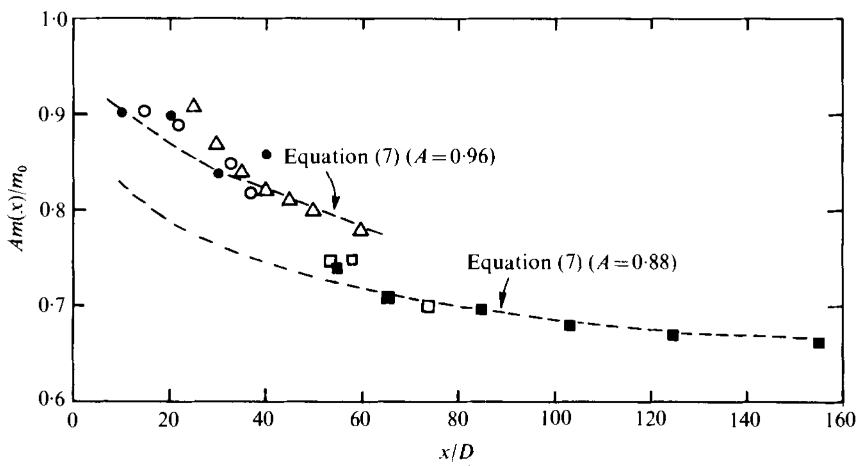

FIGURE 4. The normalized momentum flux $Am(x)/m_0$ (table 1) *vs*. the non-dimensional distance x/D. △, Goldschmidt (1964); ■, Heskestad (1965); ○, □, Kotsovinos (1975); ●, Miller (1957).

1969) or to adjust their results. Albertson *et al.* (1950) reported that the momentum flux was constant (to within 1–2 %) up to $2300D$, which contradicts the much more accurate measurements of other investigators (reported in table 1) and the analysis of this study. It therefore seems reasonable to discount the conclusion by Albertson *et al.* (1950) of constancy of the axial momentum.

Goldschmidt (1964) speculated that the boundary layers along the two confining walls (parallel to the x, y plane) are responsible for the momentum deficiency observed in his experimental results. However, these two boundary layers do not reach the mid-plane parallel to the two confining walls (where measurements are taken) for $x < 2h$–$3h$ (where h is the separation of the two confining walls), as was demonstrated by Van der Hegge Zijnen (1957) and Heskestad (1965). On the other hand it is estimated, assuming a friction coefficient c_f about 0·01, that the total force along the walls is negligible relative to the input total force $\rho m_0 h$.

Kraemer (1971) predicted that for a plane jet out of a wall

$$2\int_{B(x)}^{\infty} \overline{u}^2(x,y)\,dy = 0{\cdot}03m_0.$$

This result should not be confused with the reduction $m_0 - m(x)$ in the kinematic momentum flux, because

$$m_0 - m(x) = 2\int_{B(x)}^{\infty} \overline{u}^2(x,y)\,dy + 2\int_{B(x)}^{\infty} (\overline{p}(x,y)/\rho)\,dy - 2\int_{B(0)}^{\infty} (\overline{p}(0,y)/\rho)\,dy.$$

In Kraemer's (1971) analysis both pressure integrals diverge and therefore his analysis can not be used for realistic estimates.

5. Conclusions

A basic feature of free jets is that they induce flow towards themselves. In the particular case of a plane jet out of a wall the induced flow has a component in the direction opposite to the jet flow, which tends to reduce the momentum flux from the input value. In addition, the pressure field which builds up throughout the potential region has a positive gradient, which also tends to reduce the momentum flux. A rough estimation shows that this reduction is not negligible. An approximate model for the reduction in the momentum flux is proposed and is found to be in fair agreement with existing experimental results.

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